TA NOTES

Section 7.8

Find the general solution of the given systems of equations and describe how the solution behave as $t \longrightarrow \infty$.

3. $\mathbf{x}' = \begin{pmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} \mathbf{x}$

6. $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}$

Answer: 3. $\mathbf{A} = \begin{pmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} \mathbf{x}$, so the eigenvalues are $\lambda = -1$. For $\lambda = -1$, we have $\xi^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. For $(A + I)\eta = \xi^1$, we can get that $\eta = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$. Hence the general solution is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 t e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

6. $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, so the eigenvalues are $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 2$. For $\lambda = -1$, we have $\xi^1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \xi^2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. For $\lambda = 2$, we have $\xi^3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Hence the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}.$$

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15. Show that all solutions of the system

$$\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x}$$

approach zero as $t \to \infty$ if and only if $a + d < 0$ and $ad - bc > 0$. Compare this result with that of Problem 38 in Section 3.5.

Answer: The eigenfunction of matrix is

$$r^2 - (a+d)r + (ad - bc) = 0$$

a). If $r_1 + r_2 = a + d < 0$ and $r_1 r_2 = ad - bc > 0$, the real part of two eigenvalue are negative. Then the general solution of the system has the form

$$c_1 e^{r_1 t}\xi^1 + c_2 e^{r_2 t}\xi^2 \quad \text{or } c_1 e^{r_1 t}\xi^1 + c_2 e^{r_1 t}(t\xi^1 + \xi^2)$$

It is clear that the solutions approach zero since $r_1 < 0$ and $r_2 < 0$.

b). If all the solutions of the system approach zero, the real part of the eigenvalues must be negative. Hence $a + d = r_1 + r_2 = Re(r_1) + Re(r_2) < 0$. If there are two real eigenvalues, then $ad - bc = r_1 r_2 > 0$. If there are two conjugate complex eigenvalues, then $ad - bc = r_1 r_2 = |r_1|^2 = Re(r_1)^2 + Im(r_2)^2 > 0$.

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17. Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \mathbf{x} \qquad (i).$

(a). Show that $r = 2$ is an eigenvalue of algebraic multiplicity 3 of the coefficient matrix $\mathbf{A}$ and that there is only one corresponding eigenvector, namely,

$$\xi^1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

(b). Using the information in part (a), write down one solution $\mathbf{x}^1(t)$ of the system (i).

(c). To find a second solution assume that $\mathbf{x} = \xi t e^{2t} + \eta e^{2t}$. Show ξ and η satisfy the equations

$$(\mathbf{A} - 2\mathbf{I})\xi = 0, \quad (\mathbf{A} - 2\mathbf{I})\eta = \xi.$$

(d). To find a third solution assume that $\mathbf{x} = \xi(\frac{t^2}{2})e^{2t} + \eta t e^{2t} + \zeta e^{2t}$. Show that ξ, η, ζ satisfy

$$(\mathbf{A} - 2\mathbf{I})\xi = 0, \quad (\mathbf{A} - 2\mathbf{I})\eta = \xi, \quad (\mathbf{A} - 2\mathbf{I})\zeta = \eta.$$

(e). Write down a fundamental matrix Ψ for system (i).

(f). Form a matrix T with the eigenvector ξ^1 in the first column and the generalized eigenvector η and ζ in the second and third columns. Then find T^{-1} and form the product $J = T^{-1}\mathbf{A}T$.

Answer: (a). Since $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}$, so the eigenvalues are $\lambda = 2$. For $\lambda = 2$, the $\xi^1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

(b). $\mathbf{x}^1 = e^{2t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

(c).Let $\mathbf{x} = \xi t e^{2t} + \eta e^{2t}$, then we have $\mathbf{x}' = \xi e^{2t} + 2\xi t e^{2t} + 2\eta e^{2t} = \mathbf{A}\xi t e^{2t} + \mathbf{A}\eta e^{2t}$. Hence

$$(\mathbf{A} - 2\mathbf{I})\xi = 0, \quad (\mathbf{A} - 2\mathbf{I})\eta = \xi.$$

and we can compute that $\eta = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. So$\mathbf{x}^2 = te^{2t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(d). Let $\mathbf{x} = \xi(\frac{t^2}{2})e^{2t} + \eta t e^{2t} + \zeta e^{2t}$, using the same idea we can show that

$$(\mathbf{A} - 2\mathbf{I})\xi = 0, \quad (\mathbf{A} - 2\mathbf{I})\eta = \xi, \quad (\mathbf{A} - 2\mathbf{I})\zeta = \eta.$$

And we can compute that $\zeta = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$, and $\mathbf{x}^3 = \frac{t^2}{2}e^{2t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + te^{2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + e^{2t}\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$. So the fundamental matrix is

$$\Psi(t) = e^{2t}\begin{pmatrix} 0 & 1 & t+2 \\ 1 & t+1 & \frac{t^2}{2}+t \\ -1 & -t & -\frac{t^2}{2}+3 \end{pmatrix}.$$

$T = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}$, $T^{-1} = \begin{pmatrix} -3 & 3 & 2 \\ 3 & -2 & -2 \\ -1 & 1 & 1 \end{pmatrix}$,$J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. ❏

20. Let

$$\mathbf{J} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

where λ is an arbitrary real number.

(a) Find $\mathbf{J}^2$, $\mathbf{J}^3$, $\mathbf{J}^4$.

(b) Use an inductive argument to show that

$$\mathbf{J}^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}$$

(c) Determine $\exp(\mathbf{J}t)$.

(d) Observe that if you choose $\lambda = 1$, then the matrix $\mathbf{J}$ in this problem is the same as the matrix $\mathbf{J}$ in Problem 18(f). Using the matrix $\mathbf{T}$ from Problem 18(f), form the product $\mathbf{T}\exp(\mathbf{J}t)$ with $\lambda = 1$. Is the resulting matrix the same as the fundament matrix $\mathbf{\Psi}(t)$ in Problem 18(e)? If not, explain the discrepancy.

Answer: (a) By direct computation, we get

$$\mathbf{J}^2 = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix} \quad \mathbf{J}^3 = \begin{pmatrix} \lambda^3 & 0 & 0 \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{pmatrix} \quad \mathbf{J}^4 = \begin{pmatrix} \lambda^4 & 0 & 0 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{pmatrix}$$

(b) It is clearly that the equality is true for $n = 1$. Assume that the equality is true for $n = k$. Then, by inductive argument and direct computation

$$\mathbf{J}^{k+1} = \begin{pmatrix} \lambda^k & 0 & 0 \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^{k+1} & 0 & 0 \\ 0 & \lambda^{k+1} & (k+1)\lambda^k \\ 0 & 0 & \lambda^{k+1} \end{pmatrix}$$

(c) By definition,

$$\begin{aligned} \exp(\mathbf{J}t) &= \mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{n!}(\mathbf{J}t)^n = \mathbf{I} + \sum_{n=1}^{\infty} \frac{t^n}{n!} \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix} \\ &= \begin{pmatrix} 1 + \sum\limits_{n=1}^{\infty} \frac{t^n}{n!}\lambda^n & 0 & 0 \\ 0 & 1 + \sum\limits_{n=1}^{\infty} \frac{t^n}{n!}\lambda^n & \sum\limits_{n=1}^{\infty} \frac{t^n}{(n-1)!}\lambda^{n-1} \\ 0 & 0 & 1 + \sum\limits_{n=1}^{\infty} \frac{t^n}{n!}\lambda^n \end{pmatrix} \\ &= \begin{pmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix} \end{aligned}$$

(d) It is easy to know in Problem 18(f) that

$$\mathbf{T} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{pmatrix}, \quad \mathbf{T}^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 \\ 2 & -\frac{3}{2} & -1 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Then,

$$\mathbf{T}\exp(\mathbf{J}t) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix} = \begin{pmatrix} e^t & 2e^t & 2te^t \\ 0 & 4e^t & 4te^t \\ 2e^t & -2e^t & -(2t+1)e^t \end{pmatrix}$$

It is as the fundament matrix $\mathbf{\Psi}(t)$ in Problem 18(e).

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